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On the Hydrodynamic Limit of a One-Dimensional Ginzburg–Landau Lattice Model. The *a Priori* Bounds

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The simplest Ginzburg–Landau model with conservation law is investigated. The initial state is specified by an inhomogeneous profile of the chemical potential associated with the conserved quantity, that is, the mean spin. It is shown that the mean spin satisfies a nonlinear diffusion equation in the hydrodynamic limit. The proof is based on the nice, parabolic structure of the model. A standard perturbation technique is used.

KEY WORDS: Hydrodynamic limit; local equilibrium distributions; Ginzburg–Landau models; parabolic estimates.

1. INTRODUCTION

The goal of the procedure of the so-called hydrodynamic scaling limit is to derive the evolution equations of nonequilibrium thermodynamics from microscopic laws. A mathematical formulation of the problem goes back to Morrey[(23)] and Dobrushin.[(8)] (See also Ref. 6 for an exposition of the mathematical and some of the physical ideas. The early results of Refs. 3, 9, 21, and 27–29 are also of some historical interest.) The basic concept of this approach, the principle of local equilibrium, expresses a fairly deep, local ergodic property of the underlying microscopic dynamics. That is the reason why it has only been verified for some very special, more or less explicitly solvable models. In this paper we investigate a one-dimensional Ginzburg–Landau model with conservation law; see Refs. 15, 16, and 32 for a mathematical and physical interpretation of the model, and also for some references to the physics literature. This model is general enough in the sense that it contains a functional parameter. On the other hand, there is only one conservation law, and the conserved quantity satisfies a closed

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equation; the currents due to the rapidly oscillating, nonconservative quantities are represented by a space-time white noise, which makes life much easier. Let us remark that the equations for the conservative quantities of the mechanical models of Refs. 3 and 10 do close up only in the limit; therefore some rather explicit calculations are needed there.

We investigate an infinite system S of continuous spins $S(x)$ sitting at the points of the one-dimensional integer lattice $\mathbb{Z}$. Configurations of the system are real sequences of type $S\colon \mathbb{Z} \to \mathbb{R}$, where $\mathbb{R}$ denotes the real line. The formal Hamilton function of the model is supposed to be of the type

$$H(S) = \sum_{x \in \mathbb{Z}} \{V(S(x)) + \tfrac{1}{2}v[S(x+1) - S(x)]^2\} \tag{1.1}$$

where $v \geqslant 0$ is a constant, while $V\colon \mathbb{R} \to \mathbb{R}$ is a convex self-potential. The temporal evolution of the system is given by an infinite system of stochastic differential equations

$$\begin{aligned} dS(t, x) = {} & \tfrac{1}{2}[\mathbb{D}H(x+1, s) - 2\mathbb{D}H(x, S) + \mathbb{D}H(x-1, S)]\, dt \\ & + w(dt, x) - w(dt, x-1), \qquad x \in \mathbb{Z} \end{aligned} \tag{1.2}$$

with initial condition $S(0, x) = \sigma(x)$, where $w(t, x)$, $t \geqslant 0$, $x \in \mathbb{Z}$ is a family of independent, standard Wiener processes, and

$$\mathbb{D}H = \mathbb{D}H(x, S) = V'(S(x)) - v[S(x+1) - 2S(x) + S(x-1)] \tag{1.3}$$

denotes the gradient (functional derivative) of H, and V' is the derivative of V.

Throughout this paper we are assuming that V has three continuous derivatives, and we have some constants α and L, $0 \leqslant \alpha < 1$, such that

$$1 - \alpha \leqslant V''(x) \leqslant 1 + \alpha \tag{1.4}$$

and

$$|V'''(x)| \leqslant L \qquad \text{for all} \quad x \in \mathbb{R} \tag{1.5}$$

The meaning of (1.4) is simply $V(x) = x^2/2 + \alpha U(x)$, with $|U''(x)| \leqslant 1$. At the most crucial step of the proof a perturbative treatment will be used; there we also need that α is less than a universal constant α_0 to be estimated in Lemma 6. We are interested in the asymptotic behavior of the rescaled spin field

$$S_\varepsilon(t, \varphi) = \int \varphi(x)\, S(t/\varepsilon^2, [x/\varepsilon])\, dx \tag{1.6}$$

as the scaling parameter $\varepsilon > 0$ goes to zero. Here φ is a smooth function, and $[u]$ denotes the integer part of $u \in \mathbb{R}$.

From a purely mathematical point of view this problem is a very strange one. We have

$$\begin{aligned} dS_\varepsilon(t, \varphi) = {} & \tfrac{1}{2} \int (\Delta_\varepsilon \varphi(x)) \, V'(S_\varepsilon(t, x)) \, dx \, dt \\ & - \tfrac{1}{2} \nu \varepsilon^2 \int (\Delta_\varepsilon^2 \varphi(x)) \, S_\varepsilon(t, x) \, dx \, dt \\ & - \int (\nabla_\varepsilon \varphi(x)) \, w_\varepsilon(dt, x) \, dx \end{aligned} \tag{1.7}$$

were $S_\varepsilon(t, x) = S(t/\varepsilon^2, [x/\varepsilon])$, $\omega_\varepsilon(t, x) = \varepsilon w(t/\varepsilon^2, [x/\varepsilon])$, and

$$\begin{aligned} \nabla_\varepsilon \varphi(x) &= \varepsilon^{-1} [\varphi(x+\varepsilon) - \varphi(x)] \\ \Delta_\varepsilon \varphi(x) &= \varepsilon^{-2} [\varphi(x+\varepsilon) - 2\varphi(x) + \varphi(x-\varepsilon)] \end{aligned} \tag{1.8}$$

are te lattice approximations of step size ε to the differential operators $\partial/\partial x$ and $\partial^2/\partial x^2$, respectively. Te first observation is certainly that the martingale part of dS_ε vanishes together with the second integral as ε goes to zero. Thus, one might be led to the conclusion that $S_\varepsilon(t, \varphi)$ converges to a deterministic limit:

$$S_\varepsilon(t, \varphi) \xrightarrow[\varepsilon \to 0]{} \rho(t, \varphi) = \int \varphi(x) \, \rho(t, x) \, dx \tag{1.9}$$

and the limiting density ρ satisfies a nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left[D(\rho) \frac{\partial \rho}{\partial x} \right] \tag{1.10}$$

with diffusion coefficient $D(\rho) = V''(\rho)$; see Refs. 13 and 26 for the completely deterministic case when $\nu = 0$. We shall see, however, that this conclusion is false. For some randomly selected initial configurations we have, indeed, a deterministic limit as described above, but $D(\rho) \neq V''(\rho)$. The naive argument fails because of the singular behavior of $\Delta_\varepsilon V'(S)$. An interplay between the singular drift and the vanishing stochastic term results in a correction to the diffusion coefficient D of (1.10). The term $(\nu\varepsilon^2/2)\Delta_\varepsilon^2$ plays a distinguished role. Although there is no fourth derivative in the limiting equation (1.10), the modified diffusion coefficient also depends on

ν. On the other hand, if we replace the factor $\nu\varepsilon^2/2$ by $\nu/2$, then the limit is still a deterministic one, and the limiting density is governed by

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left[V''(\rho)\frac{\partial \rho}{\partial x}\right] - \frac{\nu}{2}\frac{\partial^4 \rho}{\partial x^4}$$

Moreover, if we replace the factor $\nu\varepsilon^2/2$ by $\nu/2$, and w_ε by $\varepsilon^{-1/2}w_\varepsilon$, then we are in a lattice approximation situation again. If $\nu > 0$, then some calculations by Funaki[17] suggest that in this case S_ε has a stochastic limit for all initial configurations satisfying some minimal integrability conditions, and the limiting field Y turns out to be the solution to the stochastic partial differential equation

$$dY = \frac{1}{2}\frac{\partial^2}{\partial x^2} V'(Y)\,dt - \frac{\nu}{2}\frac{\partial^4 Y}{\partial x^4}\,dt + \frac{\partial}{\partial x} w(dt, dx) \tag{1.11}$$

where $w(dt, dx)$ is a space-time white noise. If $\nu = 0$ and $V'(x) = x$, then all solutions to (1.11) are in fact generalized fields. Finally, as announced by Funaki,[17] Eq. (1.11), like Eq. (1.2), admits a scaling limit $t \to t/\varepsilon^2$, $x \to x/\varepsilon$, with random initial data.

To understand these phenomena, and in particular the role of the initial distribution, we have to go back to the physical interpretation of the problem; see Ref. 16 for a more detailed explanation. Since the right-hand side of Eq. (1.2) is a (discrete) divergence form also including the stochastic term, the spin S obviously satisfies a conservation law, and we have a one-parameter family of stationary measures μ_λ^0, $\lambda \in \mathbb{R}$. More exactly, μ_λ^0 is the Gibbs state with Hamilton function

$$H_\lambda^0(\sigma) = H(\sigma) - \lambda \sum_{x \in \mathbb{Z}} \sigma(x) \tag{1.12}$$

at unit temperature; thus, the parameter λ is just the chemical potential associated with the spin. It is easy to check that

$$\int \mathbb{D}H(x, \sigma)\,\mu_\lambda^0(d\sigma) = \lambda \tag{1.13}$$

$$\int \sigma(x)\,\mu_\lambda^0(d\sigma) = F'(\lambda) \tag{1.14}$$

for all $x \in \mathbb{Z}$, where F' denotes the derivative of the free energy for H_λ^0; F' is a strictly increasing function in our case.

In such situations the principle of local equilibrium suggests that the initial distribution should be specified as a family of local equilibrium states

$\mu_{\lambda,\varepsilon}$ with some smooth profile λ: $\mathbb{R} \to \mathbb{R}$ of the chemical potential. If (1.13) remains in force, at least in an asymptotic sense, for positive times with a time-dependent profile $\lambda = \lambda(t, x)$, then taking expectations of both sides of (1.7), we obtain a limiting equation for the mean spin ρ,

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{1}{2}\frac{\partial^2 \lambda}{\partial x^2}(t, x); \qquad \rho(0, x) = F'(\lambda(0, x)) \tag{1.15}$$

Therefore, the diffusion coefficient of (1.10) equals $D(\rho) = \partial\lambda(t, x)/\partial\rho(t, x)$. In view of the correspondence (1.14) between λ and ρ, we expect that D depends on t and x only trough ρ; thus, D: $\mathbb{R} \to (0, \infty)$ is defined by $D(v) = 1/F''(u)$ if $v = F'(u)$.

In the next section we formulate the main result of this paper, claiming that the situation is essentially the same as outlined above. Some extensions to arbitrary dimensions and to systems with reaction terms and driving forces (see Refs. 2 and 5) are to be discussed in a forthcoming continuation of this paper.

2. MAIN RESULT

We have emphasized that the hydrodynamic limit is very different from a lattice approximation procedure. Nevertheless, it will be convenient to rescale Eq. (1.2) according to the rules $t \to t/\varepsilon^2$ and $x \to [x/\varepsilon]$, and to embed the rescaled equations into a functional space $\mathbb{L}_e^2 = \mathbb{L}_e^2(\mathbb{R})$. The initial distributions will be rescaled in the same way, and they will be considered as probability measures on the common configuration space $\mathbb{L}_e^2$. From (1.7) we see immediately that (1.2) rescales into

$$dS_\varepsilon(t, x) = \tfrac{1}{2}\Delta_\varepsilon V'(S_\varepsilon)\, dt - \tfrac{1}{2}v\varepsilon^2 \Delta_\varepsilon^2 S_\varepsilon\, dt - \nabla_\varepsilon^* w_\varepsilon(dt, x) \tag{2.1}$$

where ∇_ε^* denotes the formal adjoint of ∇_ε, that is,

$$\nabla_\varepsilon^* \varphi(x) = (1/\varepsilon)[\varphi(x - \varepsilon) - \varphi(x)] \tag{2.2}$$

The evolution can formally be extended to all locally integrable initial configurations $\sigma \in \mathbb{L}_e^2$ by the trivial convention $S_\varepsilon(0, x) = \sigma_\varepsilon(x) = I_\varepsilon\sigma(x)$, where

$$I_\varepsilon\sigma(x) = \frac{1}{\varepsilon}\int_{k\varepsilon}^{k\varepsilon+\varepsilon} \sigma(y)\, dy \qquad \text{if} \quad [x/\varepsilon] = k \tag{2.3}$$

Of course, the time-evolved configurations depend only on the projection $\sigma_\varepsilon = I_\varepsilon\sigma$; they are step functions of class $\mathbb{L}_\varepsilon$, where $\varphi \in \mathbb{L}_\varepsilon$ means that φ: $\mathbb{R} \to \mathbb{R}$ and $\varphi(x) = \varphi([x/\varepsilon])$ for all $x \in \mathbb{R}$.

Under condition (1.4), Eq. (2.1) lives most happily in a scale of weighted $\mathbb{L}^2$ spaces with weight functions θ_r, $r \in \mathbb{R}$, defined as follows. Let $\theta: \mathbb{R} \to (0, 1]$ be a nonincreasing and twice continuously differentiable function such that $\theta(u) = \frac{1}{2}e^{2-u}$ if $u \geqslant 2$, $\theta(u) = 1$ if $u \leqslant 1$, $\theta(u) \geqslant e^{-u}$ if $u \geqslant 0$, and $0 \leqslant -\theta'(u) \leqslant \theta(u) \leqslant \frac{1}{2}e^{2-u}$ for all u. We define θ_r by $\theta_r(x) = [\theta(|x|)]^r$ for all $x \in \mathbb{R}$ and $r \in \mathbb{R}$ (see Refs. 13 and 14). Introduce now $\mathbb{L}_r^2 = \mathbb{L}_r^2(\mathbb{R})$ as the real Hilbert space of locally integrable $\sigma: \mathbb{R} \to \mathbb{R}$ with norm $|\cdot|_r$ defined by

$$|\sigma|_r^2 = \int \theta_r(x)\, \sigma^2(x)\, dx \tag{2.4}$$

The associated scalar product will be denoted as $\langle \cdot, \cdot \rangle_r$. Since $\theta_r(x) \leqslant \theta_s(x)$ if $s < r$, we have $\mathbb{L}_s^2 \subset \mathbb{L}_r^2$ in this case. Moreover, $\mathbb{L}_{-r}^2 \subset \mathbb{L}^2(\mathbb{R}) \subset \mathbb{L}_r^2$ if $r > 0$, and they are the dual spaces of each other with respect to the usual scalar product, $\langle \cdot, \cdot \rangle_0$ of $\mathbb{L}^2(\mathbb{R})$. Now we define a configuration space, $\mathbb{L}_e^2$ for (2.1) as the locally convex space with seminorms $|\cdot|_r$, $r > 0$. This simply means that

$$\mathbb{L}_e^2 = \mathbb{L}_e^2(\mathbb{R}) = \bigcap_{r>0} \mathbb{L}_r^2(\mathbb{R}) \tag{2.5}$$

and $\sigma_n \to \sigma$ in the (strong) topology of $\mathbb{L}_e^2$ iff $|\sigma_n - \sigma|_r \to 0$ for each $r > 0$. A subset $B \subset \mathbb{L}_e^2$ is bounded if $[|\sigma|_r\colon \sigma \in B]$ is bounded for each $r > 0$. The dual space $\mathbb{L}_e^{2*}$ of $\mathbb{L}_e^2$ is just

$$\mathbb{L}_e^{2*} = \mathbb{L}_e^{2*}(\mathbb{R}) = \bigcup_{r>0} \mathbb{L}_{-r}^2(\mathbb{R}) \tag{2.6}$$

The elements of $\mathbb{L}_e^{2*}$ as linear functionals will be denoted as

$$\varphi = \varphi(\sigma) = \int \varphi(x)\, \sigma(x)\, dx, \qquad \sigma \in \mathbb{L}_e^2 \tag{2.7}$$

Remember that $\mathbb{L}_e^{2*}$ is not a metric space; $\varphi_n \to \varphi$ in $\mathbb{L}_e^{2*}$ means that there exists an $r > 0$ such that $\varphi_n \to \varphi$ in $\mathbb{L}_{-r}^2$. The weak topology of $\mathbb{L}_e^2$ is not metrizable either; it is given by a fundamental system of the neighborhoods of $0 \in \mathbb{L}_e^2$, namely

$$U_\gamma(\varphi_1, \varphi_2,..., \varphi_n) = [\sigma \in \mathbb{L}_e^2\colon |\varphi_k(\sigma)| < \gamma,\ k = 1, 2,..., n] \tag{2.8}$$

where $\gamma > 0$, $n \in \mathbb{N}$, and $\varphi_k \in \mathbb{L}_e^{2*}$. A subset B of $\mathbb{L}_e^2$ is called a ball if

$$B = [\sigma \in \mathbb{L}_e^2\colon |\sigma|_r \leqslant b_r,\ r > 0] \tag{2.9}$$

with some $b_r < \infty$. It is easy to check that $\mathbb{L}_e^2$ is a reflexive space, and every ball of $\mathbb{L}_e^2$ is weakly compact (see Ref. 34). The space of differentiable $\lambda \in \mathbb{L}_e^2$

such that λ' is absolutely continuous and λ', λ'' also belong to $\mathbb{L}_e^2$ will be denoted by $\mathbb{H}_e^2$. If $\Omega \subset \mathbb{L}_e^2$, then $\mathbb{C}_b(\Omega)$ denotes the set of strongly continuous and bounded $f: \Omega \to \mathbb{R}$, while $\mathbb{C}_w(\Omega)$ is the space of the weakly continuous elements of $\mathbb{C}_b(\Omega)$.

Now we turn to the study of Eq. (2.1). In view of (1.4), the drift of (2.1) is linearly bounded and uniformly Lipschitz continuous in any of the spaces $\mathbb{L}_r^2(\mathbb{R})$, at least if $\varepsilon > 0$ is fixed. Therefore, a standard Picard–Lindelöf-type argument yields the existence and uniqueness of strong solutions to (2.1) in each $\mathbb{L}_r^2$, (see, e.g., Ref. 4); thus, we have a transition semigroup $\mathbb{P}_\varepsilon^t$,

$$\mathbb{P}_\varepsilon^t f = \mathbb{P}_\varepsilon^t f(\sigma) = \mathbb{E}[f(S_\varepsilon(t)) \mid S_\varepsilon(0) = I_\varepsilon \sigma], \qquad f \in \mathbb{C}_b(\mathbb{L}_e^2) \tag{2.10}$$

where $S_\varepsilon(t) = S_\varepsilon(t, \cdot)$. Essentially the same argument shows that $\mathbb{P}_\varepsilon^t$ is in fact a strongly continuous contraction semigroup in $\mathbb{C}_b(\mathbb{L}_e^2)$ for each $\varepsilon > 0$; its generator will be denoted by $\mathbb{G}_\varepsilon$. We are not going to enter into details of this construction problem, but the *a priori* bounds we prove in the next section are much stronger than those one usually needs to derive such qualitative results. To find a compact expression for the generator, we need a notion of functional (variational) derivatives.

Definition 1. Let $\Omega \subset \mathbb{L}_e^2$ be convex and $f \in \mathbb{C}_b(\Omega)$. We say that f has a continuous and bounded functional derivative $\mathbb{D}f$ if we have a mapping $\mathbb{D}f: \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\mathbb{D}f: \Omega \to \mathbb{L}_e^{2*}$ is a bounded, continuous function of $\sigma \in \Omega$, and if $\delta = \sigma - \bar{\sigma}$, then for all $\sigma, \bar{\sigma} \in \Omega$ we have

$$f(\sigma) - f(\bar{\sigma}) = \int_0^1 \int \delta(x)\, \mathbb{D}f(x, \bar{\sigma} + q\delta)\, dx\, dq \tag{2.11}$$

The space of such f will be denoted by $\mathbb{D}_b(\Omega)$, while $\mathbb{D}_b^2(\Omega)$ is the set of $f \in \mathbb{D}_b(\Omega)$ such that $\mathbb{D}^2 f = \mathbb{D}\mathbb{D}f = \mathbb{D}^2 f(x, x', \sigma)$ is a continuous and bounded map of Ω into $\mathbb{L}_e^{2*} \otimes \mathbb{L}_e^{2*}$. If $\Omega \subset \mathbb{L}_\varepsilon$, then a distinguishing symbol $\mathbb{D}_\varepsilon$ will be used.

Observe now that if

$$H_\varepsilon(x, \sigma) = V(\sigma(x)) + \tfrac{1}{2}\nu\{[\sigma(x+\varepsilon) - \sigma(x)]^2 + [\sigma(x-\varepsilon) - \sigma(x)]^2\} \tag{2.12}$$

denotes the energy density of σ, then the drift of (2.1) equals $\frac{1}{2}\varepsilon\Delta_\varepsilon\, \mathbb{D}_\varepsilon H(x, \cdot)(x, \sigma)$; thus, for smooth cylinder functions we have

$$\mathbb{G}_\varepsilon f(\sigma) = -\tfrac{1}{2}\varepsilon \int e^{H(x,\sigma)} \mathbb{D}_\varepsilon[e^{-H(x,\sigma)} \Delta_\varepsilon \mathbb{D}_\varepsilon f(x, \sigma)](x, \sigma)\, dx \tag{2.13}$$

Let $\mu_{\lambda,\varepsilon}$ denote the Gibbs state on $\mathbb{L}_\varepsilon$ with interaction H, temperature 1, and chemical potential $\lambda_\varepsilon = I_\varepsilon \lambda$, $\lambda \in \mathbb{H}_e^2$; that is, the conditional density of

$\sigma(x)$, given $\sigma(y)$ for $[y/\varepsilon]\neq[x/\varepsilon]$, is proportional to $\exp[-H(x,\sigma)+\lambda_\varepsilon(x)\,\sigma(x)]$ for all $x\in\mathbb{R}$ and $\sigma\in\mathbb{L}_\varepsilon$. It is easy to check that $\lambda\in\mathbb{L}_e^2$ implies $\mu_{\lambda,\varepsilon}(\mathbb{L}_\varepsilon\cap\mathbb{L}_e^2)=1$. Integrating $\mathbb{G}_\varepsilon f$ by parts, we obtain a fundamental identity

$$\int \mathbb{G}_\varepsilon f(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=\tfrac{1}{2}\iint (\Delta_\varepsilon\lambda(x))\,\mathbb{D}_\varepsilon f(x,\sigma)\,dx\,\mu_{\lambda,\varepsilon}(d\sigma) \tag{2.14}$$

On the other hand, if $g\in\mathbb{C}_w(\mathbb{L}_e^2)$, then the law of large numbers implies

$$\lim_{\varepsilon\to 0}\int g(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=g(F'(\lambda)) \tag{2.15}$$

where $F'(\lambda)(x)=F'(\lambda(x))$ [cf. (1.14)]. Similarly, if f admits a weakly continuous functional derivative, then

$$\lim_{\varepsilon\to\infty}\int \mathbb{G}_\varepsilon f(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=\tfrac{1}{2}\int \lambda''(x)\,\mathbb{D}f(x,F'(\lambda))\,dx \tag{2.16}$$

Of course, (2.15) needs a proof. There is nothing to prove if $v=0$; in the general case (2.15) reduces to

$$\lim_{\varepsilon\to 0}\int [\varphi(\sigma)-\varphi(F'(\lambda)]^2\,\mu_{\lambda,\varepsilon}(d\sigma)=0 \tag{2.17}$$

for smooth $\varphi\in\mathbb{L}_e^{2*}$. Unfortunately, I have not found any explicit reference concerning this weak law of large numbers, but the principles are well known (see Refs. 7, 18, and 22). Another method is to combine an associated stochastic dynamics[31] with the Feynman–Kac formula to conclude (2.17); this question will be discussed elsewhere.

Theorem 2. There exists an $\alpha_0>0$ such that if $\alpha\leqslant\alpha_0$, $\lambda\in\mathbb{H}_e^2$, $g\in\mathbb{D}_b^2(\mathbb{L}_e^2)$, and the initial configuration of (2.1) is distributed by $\mu_{\lambda,\varepsilon}$, then

$$\lim_{\varepsilon\to 0}\int g(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=\bar g(\lambda)\qquad\text{for}\quad \lambda\in\mathbb{H}_e^2 \tag{2.18}$$

with some $\bar g\in\mathbb{C}_b(\mathbb{H}_e^2)$ implies for $t>0$ that

$$\lim_{\varepsilon\to 0}\int \mathbb{P}_\varepsilon^t g(\sigma)\,\mu_{\lambda,\varepsilon}(d\sigma)=\bar g(\lambda(t)) \tag{2.19}$$

where $\lambda(t)=\lambda(t,x)$ denotes the solution to $F''(\lambda)\,\partial\lambda/\partial t=\frac{1}{2}\,\partial^2\lambda/\partial x^2$ with initial condition $\lambda(0)=\lambda$.

Remark 1. If $g \in \mathbb{C}_w(\mathbb{L}_e^2)$, then (2.18) reduces to (2.15) with $\tilde{g}(\lambda) = g(F'(\lambda))$. In particular, if $g(\sigma) = h(\varphi(\sigma))$, $\varphi \in \mathbb{L}_e^{2*}$, then (2.19) implies that $S_\varepsilon(t, \varphi) \to \int \varphi(x)\, \rho(t, x)\, dx$ in probability as $\varepsilon \to 0$, where ρ solves (1.10) with $\rho(0, x) = F'(\lambda(x))$.

Remark 2. The symmetric zero-range model is very similar to our Ginzburg–Landau model, at least if $v = 0$. Rost[30] obtained a strong form of the principle of local equilibrium in that case.

Remark 3. The restriction that $\alpha \leqslant \alpha_0$ seems to be technical; it is due to a brutal perturbation method we are using to derive some parabolic estimates that do not depend on the smoothness properties of the coefficients. Fabes[11] has pointed out that this restriction can certainly be removed. Another method was proposed by Guo *et al.*[20]

The proof of the theorem is based on an adaptation of the resolvent equation method of Refs. 19, 24, and 25 to the present situation, (see also Refs. 15 and 16). This extremely flexible technique reduces the proof of (2.19) to the verification of certain smoothness properties of the evolution as a function of the initial configuration. The related, merely qualitative analysis of the microscopic dynamics exploits the parabolic structure of the model to be exposed in the next section. Although the resolvent techniques seem to be applicable to all models we have in mind, the necessary smoothness properties of the dynamics fail to hold even for the simplest hyperbolic systems, such as the harmonic crystals. Therefore, if we want to understand something about an anharmonic crystal by means of similar methods, we are presumably forced again to introduce some small noise and damping to smoothen the dynamics.

3. THE PARABOLIC STRUCTURE

The basic idea of the proof is very simple. Consider the resolvent

$$f_\varepsilon(\sigma) = \int_0^\infty e^{-zt}\, \mathbb{P}_\varepsilon^t\, g(\sigma)\, dt, \qquad z > 0 \tag{3.1}$$

Then $g = zf_\varepsilon - \mathbb{G}_\varepsilon f_\varepsilon$, whence, by (2.14)

$$\begin{aligned} \int g(\sigma)\, d\mu_{\lambda,\varepsilon} &= z \int f_\varepsilon(\sigma)\, d\mu_{\lambda,\varepsilon} \\ &\quad - \tfrac{1}{2} \iint (\Delta_\varepsilon \lambda(x))\, \mathbb{D}_\varepsilon f_\varepsilon(x, \sigma)\, dx\, d\mu_{\lambda,\varepsilon} \end{aligned} \tag{3.2}$$

We shall show that both f_ε and $\mathbb{D}f_\varepsilon$ are uniformly bounded and weakly equicontinuous on the balls of $\mathbb{L}_e^2$; thus, we can pass to

$$\bar{g}(\lambda) = zf(F'(\lambda)) - \tfrac{1}{2}\int \lambda''(x)\, \mathbb{D}f(x, F'(\lambda))\, dx \tag{3.3}$$

along the very same subsequence for all $\lambda \in \mathbb{H}_e^2$. Observe now that (3.3) is just the resolvent equation for the limiting equation $F''(\lambda)\, \partial\lambda/\partial t = \frac{1}{2}\, \partial^2\lambda/\partial x^2$. It is easy to verify that $F'' > 0$ is bounded and it is bounded away from zero; consequently, (3.3) has a unique solution, and

$$f(F'(\lambda)) = \int_0^\infty e^{-zt}\, \bar{g}(\lambda(t, \cdot))\, dt \tag{3.4}$$

This means that $\mu_{\lambda,\varepsilon}(f_\varepsilon) \to f(F'(\lambda))$ for all subsequences; thus, the proof can be completed by showing that $\mu_{\lambda,\varepsilon}(\mathbb{P}_\varepsilon^t g)$ is an equicontinuous function of time.

The *a priori* bounds we need all reduce to the study of the fundamental solution p_a of some parabolic equations of the type

$$\frac{\partial u}{\partial t}(t, y) = G_a u(t, y) \tag{3.5}$$

$$G_a u(t, y) = \tfrac{1}{2}\Delta_\varepsilon(a(t, y)\, u(t, y)) - \tfrac{1}{2}\nu\varepsilon^2 \Delta_\varepsilon^2 u(t, y) \tag{3.6}$$

where $a(t, \cdot) \in \mathbb{L}_\varepsilon$ and $|1 - a(t, y)| \leqslant \alpha$ for all $t \geqslant 0$ and $y \in \mathbb{R}$. Then $p_a = p_a(s, x; t, y)$ is defined for $0 \leqslant s \leqslant t$ and $x, y \in \mathbb{R}$ as the solution to (2.5) with boundary condition $p_a(s, x; s, y) = 1/\varepsilon$ if $[x/\varepsilon] = [y/\varepsilon]$, and $p_a(s, x; s, y) = 0$ otherwise. Observe now that if f_ε and g are related by (3.1), then

$$\mathbb{D}_\varepsilon f_\varepsilon(x, \sigma) = \mathbb{E}\left[\int_0^\infty e^{-zt} \int p_a(0, x; t, y)\, \mathbb{D}_\varepsilon g(y, S_\varepsilon(t))\, dy\, dt\right] \tag{3.7}$$

where S_ε is the solution to (2.1) with initial condition $S_\varepsilon(0) = I_\varepsilon \sigma$, while

$$a(t, y) = V''(S_\varepsilon(t, y)) \tag{3.8}$$

That is why we are interested in some properties of p_a; the only information we have on a is $|1 - a| \leqslant \alpha$.

In view of the correspondence between $v \in \mathbb{L}_b$ and $u = \nabla_\varepsilon v$, the study of G_a can partly be reduced to that of L_a,

$$L_a v = -\tfrac{1}{2}\nabla_\varepsilon^*(a\nabla_\varepsilon v) - \tfrac{1}{2}\nu\varepsilon^2 \Delta_\varepsilon^2 v \tag{3.9}$$

Lemma 1. There exists a constant $z_0 > 0$ depending only on α, such that if $\partial v/\partial t = L_a v + \nabla_\varepsilon^* h$ with some $h(t) \in \mathbb{L}_e^2$ for $t \geqslant s$, then

$$\tfrac{1}{2}(1-\alpha) \int_s^\infty e^{-zt} |\nabla_\varepsilon v(t)|_r^2 \, dt$$

$$\leqslant e^{-zs} |v(s)|_r^2 + z_0 \int_s^\infty e^{-zt} |h(t)|_r^2 \, dt$$

for all $s \geqslant 0$, $z \geqslant z_0$, $|r| \leqslant 1$, and $0 < \varepsilon \leqslant 1$.

Proof. It is sufficient to show that

$$\begin{aligned} 2\langle v, L_a v\rangle_r + 2\langle v, \nabla_\varepsilon^* h\rangle_r + \tfrac{1}{2}(1-\alpha) |\nabla_\varepsilon v|_r^2 \\ \leqslant z_0 |v|_r^2 + z_0 |h|_r^2 \end{aligned} \tag{3.10}$$

Since ∇_ε^* is the adjoint of ∇_ε in $\mathbb{L}^2(\mathbb{R})$ and $-\Delta_\varepsilon = \nabla_\varepsilon^* \nabla_\varepsilon$,

$$2\langle v, L_a v\rangle_r = -\int (\nabla_\varepsilon \theta_r v) a \nabla_\varepsilon v \, dy - \varepsilon^2 \int (\Delta_\varepsilon \theta_r v) \, \Delta_\varepsilon v \, dy$$

$$\langle v, V^* h\rangle_r = \int (\nabla_\varepsilon \theta_r v) h \, dy$$

On the other hand,

$$\begin{aligned} \nabla_\varepsilon \theta_r v &= \theta_r \nabla_\varepsilon v + (\nabla_\varepsilon \theta_r) T_\varepsilon v \\ \Delta_\varepsilon \theta_r v &= \theta_r \Delta_\varepsilon v + (1/\varepsilon)(\nabla_\varepsilon \theta_r) T_\varepsilon v + (1/\varepsilon)(\nabla_\varepsilon^* \theta_r) T_\varepsilon^* v \end{aligned}$$

where $T_\varepsilon \varphi(x) = \varphi(x+\varepsilon)$ and $T_\varepsilon^* \varphi(x) = \varphi(x-\varepsilon)$; consequently,

$$2\langle v, L_a v\rangle_r + 2\langle v, \nabla_\varepsilon^* h\rangle_r + (1-\alpha) |\nabla_\varepsilon v|_r^2 + (1-\alpha)\varepsilon^2 |\Delta_\varepsilon v|_r^2$$

$$\leqslant (1+\alpha) \int |\nabla_\varepsilon \theta_r| \, |T_\varepsilon v| (|\nabla_\varepsilon v| + |h|) \, dy + 2\langle \nabla_\varepsilon v, h\rangle_r$$

$$+ (1+\alpha)\varepsilon \int (|\nabla_\varepsilon \theta_r| \, |T_\varepsilon v| + |\nabla_\varepsilon^* \theta_r| \, |T_\varepsilon^* v|) |\Delta_\varepsilon v| \, dy$$

Taking into account that $|\theta_r'(x)| \leqslant \theta_r(x)$ and $\theta_r(x) \leqslant e^{|r|} \theta_r(y)$ if $|x-y| \leqslant 1$, the statement follows by an easy application of an elementary inequality, $uv \leqslant cu^2/2 + v^2/2c$ if $c > 0$. ∎

In the rest of the paper the constant of Lemma 1 will be denoted by z_0 and $r \in [-1, 1]$ will be assumed.

Lemma 2. There exists a constant M, depending only on α, such that for all $s \geqslant 0$, $z \geqslant z_0$, $\varepsilon \leqslant 1$, and $0 < r \leqslant 1$ we have

$$\int_s^\infty e^{-zt} \int |p_a(s, x; t, y)|^2 \theta_r(y)\, dy\, dt$$

$$\leqslant \frac{M}{|r|} e^{-zs} \theta_r(x)$$

Proof. Because of the symmetry of the problem, we may and do assume that $x \geqslant 0$ and $r > 0$. Define $v_x(t) = v_x(t, y)$ as the solution to $\partial v/\partial t = L_a v$ with $v_x(s, y) = 0$ if $[y/\varepsilon] \leqslant [x/\varepsilon]$, and $v_x(s, y) = 1$ otherwise; then $\nabla_\varepsilon v_x(t, y) = p_a(x, s; t, y)$ if $t \geqslant s$; thus, taking the Laplace transform of both sides of (3.10) with $h = 0$ and $z \geqslant z_0$, we obtain the statement. ∎

In the one-dimensional case we are considering it is quite easy to define a bounded inverse of ∇_ε. Indeed, let $K_\varepsilon \colon \mathbb{L}_e^2 \to \mathbb{L}_e^2 \cap \mathbb{L}_\varepsilon$ be defined by

$$\begin{aligned} K_\varepsilon u(y) &= \int_0^{k\varepsilon} u(x)\, dx \qquad \text{if} \quad k = [y/\varepsilon] \geqslant 0 \\ K_\varepsilon u(y) &= \int_{k\varepsilon}^0 u(x)\, dx \qquad \text{if} \quad k = [y/\varepsilon] \leqslant 0 \end{aligned} \tag{3.11}$$

It is plain that $K_\varepsilon u(0) = 0$ and $\nabla_\varepsilon K_\varepsilon u = I_\varepsilon u$. Since K_ε is uniformly compact in $\mathbb{L}_e^2$, the solutions to (2.1) turn out to be weakly continuous functions of the initial data. That is why, instead of one of the more familiar spaces $\mathbb{L}_r^2$, we have to work with a full scale of spaces. This technical difficulty can be avoided by considering the problem in a bounded domain only (see Refs. 20 and 30).

Lemma 3. For each $\beta > 0$, $r \in (0, 1]$, and for each ball B of $\mathbb{L}_e^2$ we have a $\gamma > 0$ and some $\varphi_1, \varphi_2, \dots, \varphi_n \in \mathbb{L}_e^{2*}$ such that for all $s \geqslant 0$ and $z \geqslant z_0$ we have

$$\int_s^\infty e^{-zt} \left[\int \delta(x)\, p_a(s, x; t, y) \right]^2 \theta_r(y)\, dy\, dt \leqslant \beta e^{-zs}$$

whenever $\delta \in B \cap U_\gamma(\varphi_1, \varphi_2, \dots, \varphi_n)$.

Proof. Define $v_\delta = v_\delta(t)$ for $t \geqslant s$ as the solution to $\partial v/\partial t = L_a v$ with initial condition $v_\delta(s) = K_\varepsilon \delta$. In view of Lemma 1 and the Schwartz inequality, we have to estimate $|K_\varepsilon \delta|_r$. Observe that

$$|K_\varepsilon \delta|_r^2 = \int \varphi_\delta(y)\, \delta(y)\, dy \tag{3.12}$$

where $\varphi_\delta = K_\varepsilon^*(\theta_r K_\varepsilon \delta)$ and K_ε^* denotes the adjoint of K_ε in $\mathbb{L}^2$. An easy calculation yields

$$|K_\varepsilon \delta|_r \leqslant \frac{M}{r} |\delta|_{r/2}, \qquad |K_\varepsilon^* \psi|_{-r/2} \leqslant \frac{M}{r} |\psi|_{-r} \tag{3.13}$$

where M is a universal constant; consequently

$$|\varphi_\delta|_{-r/2} \leqslant (M/r)^2 |\delta|_{r/2} \tag{3.14}$$

On the other hand, $\nabla_\varepsilon^* \varphi_\delta = I_\varepsilon \theta_r K_\varepsilon \delta$; therefore

$$|\nabla_\varepsilon + \varphi_\delta|_{-r} \leqslant (M/r) |\delta|_{r/2} \tag{3.15}$$

Consider now the set $K_r(B) \subset \mathbb{L}_e^{2*}$,

$$K_r(B) = \bigcup_{\varepsilon > 0} \left[\psi \in \mathbb{L}_\varepsilon : |\psi|_{-r} + |\nabla_\varepsilon^* \psi|_{-r} \leqslant \frac{M}{r} \left(1 + \frac{M}{r} \right) b_{r/2} \right] \tag{3.16}$$

where $b_r = \sup |\delta|_r$ for $\delta \in B$, and notice that $\varphi_\delta \in K_r(B)$ if $\delta \in B$. Moreover, (3.14) and (3.15) imply the compactness criterion of F. Riesz; thus, $K_r(B)$ is precompact in the strong topology of $\mathbb{L}^2(\mathbb{R})$, and hence also in $\mathbb{L}^2_{-r/2}(\mathbb{R})$. Indeed, let $\psi \in K_r(B)$ and estimate $|\psi|^2_{-r/2}$ separately in the interval $[-2n/r, 2n/r]$ and outside of this interval; we obtain

$$|\psi|^2_{-r/2} \leqslant e^n \int \psi^2 \, dy + 5e^{-n} |\psi|^2_{-r} \leqslant C_r(B) \left(\int \psi^2 \, dy \right)^{1/2} \tag{3.17}$$

Therefore, for each $\gamma > 0$ we can select a finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ from $K_r(B)$ in such a way that $|\varphi_\delta - \varphi_k|_{-r/2} < \gamma$ for each $\delta \in B$ with some $k = 1, 2, \ldots, n$. If $\delta \in B \cap U_\gamma(\varphi_1, \ldots, \varphi_n)$, then

$$\begin{aligned} \left| \int \varphi_\delta(y)\, \delta(y)\, dy \right| &\leqslant \int |\varphi_\delta(y) - \varphi_k(y)|\, |\delta(y)|\, dy + |\varphi_k(\delta)| \\ &\leqslant \gamma(1 + b_{r/2}) \end{aligned}$$

which completes the proof. ∎

This result is sufficient to prove the equicontinuity of f_ε.

4. THE PERTURBATION TECHNIQUE

To estimate a difference like $\mathbb{D}f_\varepsilon(x, \sigma) - \mathbb{D}f_\varepsilon(x, \bar{\sigma})$, we have to compare two equations of type (3.5) with different coefficients a and $\bar{a}$. We shall make use of the backward equation

$$\partial u/\partial s + G_a^* u + h = 0 \tag{4.1}$$

$$G_a^* u(s, x) = \tfrac{1}{2} a(s, x)\, \Delta_\varepsilon u(s, x) - \tfrac{1}{2} \nu \varepsilon^2 \Delta_\varepsilon^2 u(s, x) \tag{4.2}$$

where $s \geqslant 0$, $x \in \mathbb{R}$, and h: $\mathbb{R}_+^2 \to \mathbb{R}$, $\mathbb{R}_+^2 = [0, \infty) \times \mathbb{R}$. Since p_a satisfies (4.1) with $h = 0$,

$$P_a^T h = P_a^T h(s, x) = \int_s^T \int p_a(s, x; t, y)\, h(t, y)\, dy\, dt \tag{4.3}$$

solves (4.1) with boundary condition $u(T, \cdot) = 0$. Define $R_{a,\bar{a}}$ by

$$R_{a,\bar{a}} u(s, x) = \tfrac{1}{2}[a(s, x) - \bar{a}(s, x)]\, \Delta_\varepsilon u(s, x) \tag{4.4}$$

Then (4.3) implies an identity,

$$P_a^T h - P_{\bar{a}}^T h = P_{\bar{a}}^T R_{a,\bar{a}} P_{\bar{a}}^T h \tag{4.5}$$

Since the right-hand side of (4.5) is a product of three factors, $\mathbb{L}^2$-estimates are not sufficient to bound $P_a^T - P_{\bar{a}}^T$; we also need bounds for some powers $q > 2$. We have no information on the smoothness of a; thus, we are forced to use the simplest perturbation method, a Neumann expansion. We follow Chapter 9 and the Appendix of Ref. 33.

Suppose that $\bar{a} = 1$; the corresponding objects will be denoted by p_1 and P_1^T. Of course, $p_1(s, x; t, y) = p_\varepsilon(t - s,\ y - x)$,

$$\begin{aligned} p_\varepsilon(t, y) &= \frac{1}{\varepsilon} J_{[y/\varepsilon]}^{(\varepsilon)}(t/\varepsilon^2) \\ J_n^{(\varepsilon)}(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-ik_\varepsilon(\omega)] \cos n\omega\, d\omega \end{aligned} \tag{4.6}$$

where

$$k_\varepsilon(\omega) = (1 - \cos \omega)[1 + v\varepsilon^2(1 - \cos \omega)]$$

We shall consider P_a^T as a small perturbation to P_1^T.

Lemma 4. If $1 \leqslant q < \infty$, then for $t > 0$ and $\varepsilon > 0$

$$\int |p_\varepsilon(t, y)|^q\, dy \leqslant C_q t^{(1-q)/2}$$

where C_q depends only on q.

Proof. The Hausdorff–Young inequality implies that if $q \geqslant 2$ and $q' = q/(q - 1)$, then

$$\sum_{n=-\infty}^{\infty} |J_n^{(\varepsilon)}(t)|^q \leqslant C_1^q \left\{ \int \exp[-q' k_\varepsilon(\omega)]\, d\omega \right\}^{q/q'}$$

Since $k_\varepsilon(\omega) \geqslant \omega^2/5$ and we can multiply both sides by ε^{1-q}, this proves Lemma 4 for $q \geqslant 2$.

On the other hand, $p_\varepsilon(t, \cdot) = p_\varepsilon(t/2, \cdot) * p_\varepsilon(t/2, \cdot)$; thus, $|p_\varepsilon(t, y)| \leqslant C'_1 t^{-1/2}$ for all y, which completes the proof. ∎

Lemma 5. For each $q \in (1, \infty)$ we have a constant $\bar{\alpha}_q > 0$ such that $\bar{\alpha}_q$ is strictly positive in the interior of $(1, \infty)$, and for all $\alpha \leqslant \bar{\alpha}_q$ and $0 < \varepsilon \leqslant 1$, $h: \mathbb{R}^2_+ \to \mathbb{R}$,

$$|R_{a,1} P_1^\infty h|_{q+} \leqslant (1 - \bar{\alpha}_q) |h|_{q+}$$

where $|\cdot|_{q+}$ denotes the usual norm of $\mathbb{L}^q(\mathbb{R}^2_+)$.

Proof. We have to show that

$$|\Delta_\varepsilon P_1^\infty h|_{q+} \leqslant C_q |h|_{q+} \tag{4.7}$$

where C_q is bounded in the interior of $(1, \infty)$. Since $\Delta_\varepsilon p_\varepsilon(t, y)$ has a bounded Fourier transform, the case of $q = 2$ is a direct consequence of the Plancherel equality. Following the proof of Theorem A.1.6 of Ref. 33, we see that the case of $q > 2$ reduces to

$$\sup_{\delta > 0} \sup_{(s,x) \in Q_\delta} \int_{Q_{2\delta}^c} \int |\Delta_\varepsilon p_\varepsilon(t-s, y-x) - \Delta_\varepsilon p_\varepsilon(t, y)| \, dy \, dt < +\infty \tag{4.8}$$

where

$$Q_\delta = [(s, x): 0 \leqslant s < \delta^2, |x| < \delta]$$

and $Q^c_\delta = \mathbb{R}^2_+ \backslash Q_{2\delta}$. Since $\Delta_\varepsilon P_1^\infty$ is symmetric, for $q < 2$ a simple duality argument can be used. The proof of (4.8) is a question of some explicit calculations [cf. (4.6)].

The integral of (4.8) will be split into five parts. Integrating by parts and using $\omega^2/5 \leqslant 1 - \cos \omega \leqslant \omega^2/2$, we see that

$$|\Delta_\varepsilon p_\varepsilon(t, y)| \leqslant C t^{-1/2} (1 + y^2)^{-1} \tag{4.9}$$

which yields a uniform bound for the integral of (4.8) over $t < 4\delta^2$ and $|y| \geqslant 2\delta$. Similarly, we have

$$\left| \frac{\partial}{\partial t} \Delta_\varepsilon p_\varepsilon(t, y) \right| \leqslant C t^{-3/2} \min\left[\frac{1}{t}, (1 + y^2)^{-1} \right] \tag{4.10}$$

$$\begin{aligned} |\nabla_\varepsilon \Delta_\varepsilon p_\varepsilon(t, y)| &\leqslant \frac{C}{t} \min\left[\frac{1}{t}, (1 + y^2)^{-1} \right] \\ &\leqslant C t^{-5/4} (1 + y^2)^{-3/4} \end{aligned} \tag{4.11}$$

Thus, estimating $\Delta_\varepsilon p_\varepsilon(t-s, y-x) - \Delta_\varepsilon p_\varepsilon(t, y-x)$ and $\Delta_\varepsilon p_\varepsilon(t, y-x) - \Delta_\varepsilon p_\varepsilon(t, y)$ separately for $|y| \leqslant 2\delta$ and for $|y| > 2\delta$, we obtain (4.8) for the whole domain of integration, which completes the proof of Lemma 5. ∎

As a consequence, the operator

$$(I - R_{a,1} P_1^T)^{-1} = \sum_{n=0}^{\infty} (R_{a,1} P_1^T)^n$$

makes sense, and it is bounded in each $\mathbb{L}^q(\mathbb{R}_+^2)$; moreover,

$$P_a^T h = P_1^T (I - R_{a,1} P_1^T)^{-1} h \tag{4.12}$$

Taking into account Lemma 2, we also obtain an $\mathbb{L}_e^q$ bound for p_a.

Lemma 6. Let $\alpha_0 = \min \bar{\alpha}_p$ for $p \in (3/2, 2]$ and suppose that $\alpha \leqslant \alpha_0$, $z > z_0$, $2 \leqslant q < 3$, $0 < \varepsilon \leqslant 1$, and $0 < r \leqslant r_q$; then

$$\int_s^\infty e^{-zt} \int |p_a(s, x; t, y)|^q \, \theta_r(y) \, dy \, dt \leqslant C_q(r, z) e^{-zs} \theta_r(x)$$

for all $s \geqslant 0$ and $x \in \mathbb{R}$, where $r_q > 0$ and $C_q(r, z) < \infty$ depend only on q and q, r, z, respectively.

Proof. Lemma 4, Lemma 5, and the Young inequality imply that

$$\int_s^T \int p_a(s, x; t, y) \, h(t, y) \, dy \, dt \leqslant C_1 (T-s)^\gamma |h|_{p+} \tag{4.13}$$

for all $s \geqslant 0$ and $x \in \mathbb{R}$ with some $\gamma > -1$ and $C_1 < +\infty$ depending only on α_0 and p, provided that $p > 3/2$. Multiplying both sides by $(z - z_0) \exp(z_0 T - zT)$ and integrating for $T > s$, we obtain

$$\int_s^\infty \exp(z_0 t - zt) \int p_a(s, x; t, y) \, h(t, y) \, dy \, dt$$
$$\leqslant C_2 \exp(z_0 s - zs) |h|_{p+} \tag{4.14}$$

where $z > z_0$ and C_2 depends also on z. Let $I_{s,x}(r, z, q)$ denote the left-hand side of the inequality we have to prove, and consider a function h: $\mathbb{R}_+^2 \to \mathbb{R}$ defined for $t \geqslant s$ by

$$\begin{aligned} |h(t, y)| &= \exp(-z_0 t) \, |p_a(s, x; t, y)|^{q/p} \, \theta_r(y) \\ \operatorname{sign} h(t, y) &= \operatorname{sign} p_a(s, x; t, y) \end{aligned} \tag{4.15}$$

while $h(t, y) = 0$ if $t < s$. Here $s \geqslant 0$, $x \in \mathbb{R}$, $p > 3/2$, $q \geqslant 2$, and $0 \leqslant r \leqslant 1$ are arbitrary parameters. Substituting h into (4.14), we obtain that

$$I_{s,x}(r, z, 1 + q/p) \leqslant C_2 \exp(z_0 s - zs) [I_{s,x}(pr, pz_0, q)]^{1/p} \tag{4.16}$$

Consequently, Lemma 2 implies the statement for $q < 7/3$ and $r \leqslant 2/3$. Since the sequence $q_{n+1} = 1 + 2q_n/3$, $q_0 = 2$, converges to 3, iterating this argument as many times as necessary, we obtain the statement for all $q < 3$ with some $r_q > 0$. ∎

Now we are in a position to complete the proof.

5. PROOF OF THE THEOREM

We follow the resolvent approach as outlined at the beginning of Section 3. Since g is bounded, so is f_ε, while a uniform $\mathbb{L}_e^{2*}$-bound of $\mathbb{D}_\varepsilon f_\varepsilon$ follows from (3.7) by Lemma 2. Comparing Lemma 3 and (3.7) with the definition of $\mathbb{D}_\varepsilon f_\varepsilon$, we see also that f_ε is uniformly equicontinuous in the weak topology of each ball B of $\mathbb{L}_e^2$, at least if $z > z_0$. The proof of the equicontinuity of $\mathbb{D}_\varepsilon f_\varepsilon$ is a little bit more involved. Introduce the operators P_a,

$$P_a h = P_a h(s, x) = \int_s^\infty e^{-zt} \int p_a(s, x; t, y)\, h(t, y)\, dy\, dt \tag{5.1}$$

for $z > z_0$, an associated system of Hilbert norms $\|\cdot\|_{r,z}$ is defined by

$$\|h\|_{r,z}^2 = \int_0^\infty e^{-zt} \int |h(t, y)|^2\, \theta_r(y)\, dy\, dt \tag{5.2}$$

Let $a = a(t, y) = V''(S_\varepsilon(t, y))$, $\bar{a} = \bar{a}(t, y) = V''(\bar{S}_\varepsilon(t, y))$, $h(t, y) = \mathbb{D}_\varepsilon g(y, S_\varepsilon(t))$, and $\bar{h}(t, y) = \mathbb{D}_\varepsilon g(y, \bar{S}_\varepsilon(t))$, where S_ε and $\bar{S}_\varepsilon$ denote the solutions to (2.1) with initial configurations $I_\varepsilon \sigma$ and $I_\varepsilon \bar{\sigma}$, respectively; then

$$\begin{aligned} &\mathbb{D}_\varepsilon f_\varepsilon(x, \sigma) - \mathbb{D}_\varepsilon f_\varepsilon(x, \bar{\sigma}) \\ &\quad = \mathbb{E} P_a h(0, x) - \mathbb{E} P_{\bar{a}} \bar{h}(0, x) \\ &\quad = \mathbb{E} P_a (h - \bar{h})(0, x) + \mathbb{E}(P_a - P_{\bar{a}})\, \bar{h}(0, x) \end{aligned} \tag{5.3}$$

$$\begin{aligned} &h(t, y) - \bar{h}(t, y) \\ &\quad = \int_0^1 \int [S_\varepsilon(t, y') - \bar{S}_\varepsilon(t, y')]\, \mathbb{D}_\varepsilon^2 g(y, y', S_\varepsilon^{(q)}(t))\, dy'\, dq \end{aligned} \tag{5.4}$$

$$S_\varepsilon(t, y') - \bar{S}_\varepsilon(t, y') = \int \delta(x)\, p_{\bar{a}}(0, x; t, y')\, dx \tag{5.5}$$

where

$$S_\varepsilon^{(q)}(t) = qS_\varepsilon(t) + (1-q)\,\bar{S}_\varepsilon(t), \qquad \delta = \sigma - \bar{\sigma}$$
$$\tilde{a}(t, y) = [V'(S_\varepsilon(t, y)) - V'(\bar{S}_\varepsilon(t, y)][S_\varepsilon(t, y) - \bar{S}_\varepsilon(t, y)]^{-1} \tag{5.6}$$

On the other hand, if $\partial v/\partial t = L_a v + \frac{1}{2}\nabla_\varepsilon^*(a-\bar{a})p_{\bar{a}}$ with initial condition $v(0)=0$, then Lemmas 1, 2, and 6 imply

$$\begin{aligned}
&\| p_a(0, x; \cdot, \cdot) - p_{\bar{a}}(0, x; \cdot, \cdot)\|_{r,z} \\
&\quad\leqslant M\,\|(a-\bar{a})\, p_{\bar{a}}(0, x; \cdot, \cdot)\|_{r,z} \\
&\quad\leqslant M[\|(a-\bar{a})^5\|_{r,z}]^{1/10}[\|p_{\bar{a}}(0, x; \cdot, \cdot)^{5/4}\|_{r,z}]^{2/5} \\
&\quad\leqslant M_1\theta_r(x)(\|S_\varepsilon - \bar{S}_\varepsilon\|_{r,z})^{1/10}
\end{aligned} \tag{5.7}$$

provided that $z > z_0$ and $0 < r \leqslant r_{5/2}$. Since $\|S_\varepsilon - \bar{S}_\varepsilon\|_{r,z}$ can be estimated by Lemma 3, we see that $\mathbb{D}_\varepsilon f_\varepsilon$ is a uniformly equicontinuous map of each ball B of $\mathbb{L}_e^2$ into $\mathbb{L}_e^{2*}$. Lemma 2 implies also that the image of each B is compact in the weak topology of $\mathbb{L}_e^{2*}$. As a consequence, for each increasing sequence B_n of balls of $\mathbb{L}_e^2$ we can select a subsequence $\varepsilon_m \to 0$ in such a way that both f_ε and $\mathbb{D}f_\varepsilon$ converge uniformly on each B_n along this sequence. If $z > z_0$, then Lemma 2 yields a bound even for $\partial\mathbb{D}_\varepsilon f_\varepsilon/\partial z$; consequently, our subsequence can be selected independently of $z > z_0$. Finally, if f denotes the limit point of f_ε, then $\mathbb{D}_\varepsilon f_\varepsilon$ must converge to $\mathbb{D}f$, and $f \in \mathbb{D}_b(\Omega)$ with $\Omega = \bigcup B_n$.

In order to pass to the limiting resolvent equation (3.3), we need some information on the initial distributions $\mu_{\lambda,\varepsilon}$, which can be derived, e.g., by means of the auxiliary process $\omega_\varepsilon(t) = \omega_\varepsilon(t, y)$ defined by

$$\begin{aligned}
&d\omega_\varepsilon(t, y) = \tfrac{1}{2}[\lambda_\varepsilon(y) - V'(\omega_\varepsilon)]\,dt + \tfrac{1}{2}\nu\varepsilon^2\Delta_\varepsilon\omega_\varepsilon\,dt + w_\varepsilon(dt, y) \\
&\omega_\varepsilon(0, y) \in \mathbb{L}_\varepsilon \cap \mathbb{L}_e^2
\end{aligned} \tag{5.8}$$

where $\lambda_\varepsilon = I_\varepsilon\lambda$. It is easy to check that $\mu_{\lambda,\varepsilon}$ is a stationary measure of the diffusion process defined by (5.8) in $\mathbb{L}_e^2$ (see Ref. 31). Following the proof of Lemma 1, but using $|\nabla_\varepsilon\theta_r| \leqslant r\theta_r$, we obtain that

$$\frac{\partial}{\partial t}\mathbb{E}|\omega_\varepsilon(t)|_r^2 + \left(\frac{1-\alpha}{2} - rM\right)|\omega_\varepsilon(t)|_r^2 \leqslant M\left(\frac{1}{r} + |\lambda|_r^2\right) \tag{5.9}$$

Therefore, if $0 < r \leqslant (1-\alpha)/4M$, then

$$\int |\sigma|_r^2\mu_{\lambda,\varepsilon}(d\sigma) \leqslant \frac{4M}{1-\alpha}\left(\frac{1}{r} + |\lambda|_r^2\right) \tag{5.10}$$

Let $\bar{\omega}_\varepsilon$ denote the solution to (5.8) with $\bar{\lambda}_\varepsilon$ in place of λ_ε and set $\delta_\varepsilon(t, y) = [\omega_\varepsilon(t, y) - \bar{\omega}_\varepsilon(t, y)]^2$; an elementary calculation yields

$$\frac{\partial}{\partial t} \delta_\varepsilon(t, y) + \frac{1-\alpha}{2} \delta_\varepsilon(t, y)$$
$$\leqslant \frac{1}{2-2\alpha} [\lambda_\varepsilon(y) - \bar{\lambda}_\varepsilon(y)]^2 + \frac{v}{2} \varepsilon^2 \Delta_\varepsilon \delta_\varepsilon(t, y) \tag{5.11}$$

By means of a compactness argument, it is possible to show that there is a joint distribution $\mu_{\lambda,\bar{\lambda},\varepsilon}$ for $\omega_\varepsilon(0)$ and $\bar{\omega}_\varepsilon(0)$ such that $\mu_{\lambda,\bar{\lambda},\varepsilon}$ is a stationary state of the coupled process, and its marginals are just $\mu_{\lambda,\varepsilon}$ and $\mu_{\bar{\lambda},\varepsilon}$; consequently

$$(1-\alpha)\, d_{\lambda,\bar{\lambda},\varepsilon}(y) \leqslant \frac{1}{1-\alpha} [\lambda_\varepsilon(y) - \bar{\lambda}_\varepsilon(y)]^2 + v\varepsilon^2 \Delta_\varepsilon d_{\lambda,\bar{\lambda},\varepsilon}(y)$$
$$d_{\lambda,\bar{\lambda},\varepsilon}(y) = \iint [I_\varepsilon \sigma(y) - I_\varepsilon \bar{\sigma}(y)]^2\, \mu_{\lambda,\bar{\lambda},\varepsilon}(d\sigma, d\bar{\sigma}) \tag{5.12}$$

for all y . Observe that (5.12) is an inequality of type $u_m \leqslant v_m + q u_{m-1} + q u_{m+1}$, $m \in \mathbb{Z}$, with $q = v/(2v + 1 - \alpha) < 1/2$, which can be solved explicitly by iteration. We obtain that

$$u_m \leqslant \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} \binom{n}{k} v_{m-n+2k}, \qquad m \in \mathbb{Z}$$

provided that u_m and v_m increase more slowly than exponentially. This means that if v goes to zero in a dominated way, then so does u for each $m \in \mathbb{Z}$. In particular, if $\bar{\lambda}_\varepsilon(y) = \lambda(x)$ for all y, then we obtain that

$$\lim_{\varepsilon \to 0} \int \sigma(x)\, \mu_{\lambda,\varepsilon}(d\sigma) = F'(\lambda(x)) \tag{5.13}$$

for each x, and a similar statement follows for the second moments as well as for the correlations. Since

$$\sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} \binom{n}{k} e^{2ik\omega - in\omega} = \frac{1}{1 - 2q \cos \omega} \tag{5.14}$$

fairly explicit estimates are available on the local equilibrium behavior of $\mu_{\lambda,\varepsilon}$. Finally, we apply the one-dimensional version of the stochastic dynamics (5.8) to the one-dimensional conditional distributions of $\mu_{\lambda,\varepsilon}$; $d\omega$ is given by (5.8) if $[y/\varepsilon] = [x/\varepsilon]$ and $d\omega = 0$ otherwise; we see that (5.12) turns into Dobrushin's uniqueness condition. Therefore, $\mu_{\lambda,\varepsilon}$ obeys an exponential decay of correlations implying the law of large numbers (see Ref. 22).

In view of (5.10), we can choose a sequence of balls B_n in such a way that for each ball B' and $\gamma < 1$ we have an $n_0 < \infty$ such that $\mu_{\lambda,\varepsilon}(B_n) > \gamma$ if $n > n_0$ and $\lambda \in B'$. On the other hand, the law of large numbers mentioned above implies (2.15) and (2.16); thus, we can really pass to (3.3). Since F'' is a smooth and bounded function, and it is bounded away from zero, (3.3) has a unique solution in $\mathbb{D}_b(\Omega)$, $\Omega = \bigcup B_n$; consequently

$$\lim_{\varepsilon \to 0} \int_0^\infty e^{-zt} \mathbb{P}_\varepsilon^t g(\sigma)\, \mu_{\lambda,\varepsilon}(d\sigma) = \int_0^\infty e^{-zt} \bar{g}(\lambda(t))\, dt \tag{5.15}$$

for $z > z_0$, where $2F''(\lambda)\, \partial\lambda/\partial t = \partial^2\lambda/\partial x^2$ with $\lambda(0) = \lambda$. Finally since

$$\mathbb{P}_\varepsilon^{t+s} g - \mathbb{P}_\varepsilon^t g = \int_t^{t+s} \mathbb{G}_\varepsilon \mathbb{P}_\varepsilon^q g\, dq$$

Lemma 2 implies that $\mu_{\lambda,\varepsilon}(\mathbb{P}_\varepsilon^t g)$ is an equicontinuous function of time. This means that (5.15) is possible only if $\mu_{\lambda,\varepsilon}(\mathbb{P}_\varepsilon^t g)$ conwerges to $\bar{g}(\lambda(t))$, which completes the proof of the theorem. ∎

This proof is much more general than it seems to be.

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